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# PROPAGATION OF CONCENTRATION/DENSITY DISTURBANCES IN AN INERTIALLY COUPLED TWO-PHASE DISPERSION

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**Abstract**—Geurst's equations are used to predict the speed of disturbance wave propagation in a mixture containing a compressible dispersed phase. Results are obtained for the case when there is no relative velocity ahead of the disturbance and are compared with Karplus' data for air–water mixtures. The changes in density, void fraction and the velocities of each phase across the wave are predicted.

*Key Words:* exertia, inertial coupling, dispersion, wave propagation, finite waves

## SHOCK CONDITIONS

We start with the macroscopic equations of motion for the potential flow of a two-phase dispersion derived by Wallis (1989a–c) by the method of *progressive examples*, which is actually a version of the space-averaging procedure, and by Geurst (1985a,b, 1986) by the *variational method.*

The continuity equations for the continuous and dispersed phase, respectively, are as follows:

$$\frac{\partial}{\partial t}(\rho_1\epsilon_1) + \nabla\cdot(\rho_1\epsilon_1\mathbf{v}_1) = 0 \qquad [1]$$

and

$$\frac{\partial}{\partial t}(\rho_2\epsilon_2) + \nabla\cdot(\rho_2\epsilon_2\mathbf{v}_2) = 0; \qquad [2]$$

$$(\epsilon_1 + \epsilon_2 = 1).$$

We consider the momentum conservation equations for the combined phases, using the momentum density and combined momentum flux and stress tensor, and for the fluid alone. The most suitable form of these equations (Wallis 1989a,b) for the purpose of the present work is as follows (there is no requirement that the dispersed phase be incompressible):

$$\frac{\partial}{\partial t}(\rho_1\epsilon_1\mathbf{v}_1 + \rho_2\epsilon_2\mathbf{v}_2) + \nabla\cdot(\rho_1\epsilon_1\mathbf{v}_1\mathbf{v}_1 + \rho_2\epsilon_2\mathbf{v}_2\mathbf{v}_2 + \rho_1\epsilon_1 E\mathbf{w}\mathbf{w}) + \nabla(\epsilon_1 p_1 + \epsilon_2 p_2) = 0 \qquad [3]$$

and

$$\frac{\partial}{\partial t}(\mathbf{v}_1 + E\mathbf{w}) + \nabla\left(\mathbf{v}_1\cdot(\mathbf{v}_1 + E\mathbf{w}) - \tfrac{1}{2}v_1^2 - \tfrac{1}{2}Ew^2\right) - \mathbf{v}_1\times\nabla\times(\mathbf{v}_1 + E\mathbf{w}) + \frac{\nabla p_1}{\rho_1} = 0. \qquad [4]$$

Here $E(\epsilon_2)$ is the *exertia* defined by Wallis (1989c); terms including $E$ are responsible for the interphase interaction due to inertial effects associated with the relative motion of the continuous and dispersed phases; $\mathbf{w}$ is the relative velocity, $\mathbf{v}_1 - \mathbf{v}_2$. The closure of the system [1]–[4] is given by the equation of state of the dispersed phase (which we specify below) and the pressure difference between the phases (Wallis 1989c; 1991, this issue, pp. 683–695):

$$p_1 - p_2 = \tfrac{1}{2}\rho_1\epsilon_1 w^2\frac{\mathrm{d}E}{\mathrm{d}\epsilon_2}. \qquad [5]$$

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External forces per unit volume of a certain phase are henceforth omitted; such forces do not influence the considered propagation of concentration waves in a two-phase medium (a special case of so-called "surface external forces" is withdrawn from consideration). We note that the traditionally used fluid–particle viscous–drag force is not included in the present model derived for the suspension of particles in an ideal fluid. The common phenomenological approach based on the combination of viscous and inertial effects in interphase interaction can, in principle, be used for further development of the model under consideration.

Now we consider a surface of discontinuity in fluid properties. Of course, all hydrodynamic parameters (the phase concentration, the density of the dispersed phase, velocities and pressures of the continuous and dispersed phases) have jumps across the considered surface.

Although we shall consider one-dimensional waves in this work, it is useful to derive a general form of small shock conditions in the three-dimensional case. We assume that the flow of the continuous phase is irrotational. Taking into account that (Wallis 1991)

$$\nabla \times \mathbf{v}_1 = -\nabla \times (E\mathbf{w}), \qquad [6]$$

we obtain [1]–[4] in the gradient form, which immediately leads to the jump conditions:

$$[\epsilon_1 v_{1n}] = 0, \quad [\rho_2 \epsilon_2 v_{2n}] = 0, \qquad [7]$$

$$[\rho_1 \epsilon_1 v_{1n}^2 + \rho_2 \epsilon_2 v_{2n}^2 + \rho_1 \epsilon_1 E w_n^2 + \epsilon_1 p_1 + \epsilon_2 p_2] = 0 \qquad [8]$$

and

$$\left[\tfrac{1}{2} v_1^2 + E\mathbf{v}_1 \cdot \mathbf{w} - \tfrac{1}{2} E w^2 + \frac{p_1}{\rho_1}\right] = 0, \qquad [9]$$

where $[A]$ denotes the jump of a value $A$ across the small shock. The subscript n denotes the velocity component normal to the discontinuity surface. We note that the shock conditions at $\nabla \times \mathbf{v}_1 \neq 0$ are important only in the case of contact discontinuities.

Now we consider only one-dimensional flow when the condition $\nabla \times \mathbf{v}_1 = 0$ is satisfied automatically. We assume that the dispersed phase is barotropic, such that

$$p_2 = p_2(\rho_2). \qquad [10]$$

We note that if [10] is not valid, it is necessary to take into detailed consideration heat- and mass-transfer processes between the dispersed and continuous phase (Nigmatulin 1978). In this work we intend to consider only purely mechanical phenomena focusing on the effects of inertial coupling of the phases.

We denote the (macroscopic) hydrodynamic parameters in front of the discontinuity by the superscript "0". An absence of the superscript indicates hydrodynamic values behind the shock. Using [5] we get the one-dimensional shock conditions:

$$\epsilon_1 v_1 = \epsilon_1^0 v_1^0, \qquad [11]$$

$$\rho_2 \epsilon_2 v_2 = \rho_2^0 \epsilon_2^0 v_2^0, \qquad [12]$$

$$\rho_1 \epsilon_1 v_1^2 + \rho_2 \epsilon_2 v_2^2 + \rho_1 H(\epsilon_2) w^2 + p_2(\rho_2) = \rho_1 \epsilon_1^0 (v_1^0)^2 + \rho_2^0 \epsilon_2^0 (v_2^0)^2 + \rho_1 H(\epsilon_2^0)(w^0)^2 + p_2(\rho_2^0), \qquad [13]$$

and

$$\rho_1(\tfrac{1}{2} v_1^2 + E(\epsilon_2) v_1 w + \tfrac{1}{2} F(\epsilon_2) w^2) + p_2(\rho_2) = \rho_1(\tfrac{1}{2}(v_1^0)^2 + E(\epsilon_2^0) v_1^0 w^0 + \tfrac{1}{2} F(\epsilon_2^0)(w^0)^2) + p_2(\rho_2^0); \qquad [14]$$

$$(\epsilon_1 + \epsilon_2 = 1).$$

Here we introduced the functions

$$H(\epsilon_2) = \tfrac{1}{2} \epsilon_1^2 \frac{dE}{d\epsilon_2} + \epsilon_1 E \qquad [15a]$$

and

$$F(\epsilon_2) = \epsilon_1 \frac{dE}{d\epsilon_2} - E. \qquad [15b]$$

In the case of a barotropic dispersed phase (e.g. gas bubbles at isothermal conditions) the system [11]–[14] is sufficient to find all hydrodynamic parameters behind the shock from the given parameters in front of it.

## A WEAK SHOCK PROPAGATING IN A MOTIONLESS TWO-PHASE MEDIUM

We start with the case when there is no relative motion in the two-phase medium in front of the shock, so that $w^0 = 0$. Choosing the appropriate coordinate system we assume that this two-phase medium is motionless in front of the shock. The considered situation seems to be the simplest one; nevertheless, it is shown below that the wave velocity is high enough (of the order of several tens of m/s) that the present consideration covers situations when the relative velocity is reasonably high (of the order of several m/s) in front of the shock.

In the coordinate system connected with the shock

$$v_1^0 = v_2^0 = -D, \qquad [16]$$

where $D$ is the velocity of shock propagation.

Now we consider the propagation of a weak shock. Assuming small deviations of the hydrodynamic values behind the shock from the parameters in front of it:

$$\epsilon_1 = \epsilon_1^0 - \epsilon^1, \qquad \epsilon_2 = \epsilon_2^0 + \epsilon^1, \qquad \rho_2 = \rho_2^0 + \rho_2^1,$$

$$v_1 = -D + v_1^1, \qquad v_2 = -D + v_2^1, \qquad (w = v_1^1 - v_2^1),$$

$$p_2 = p_2^0 + \left(\frac{\mathrm{d}p_2}{\mathrm{d}\rho_2}\right)_{\rho_2 = \rho_2^0} \rho_2^1, \qquad [17]$$

where $A^1 \ll A^0$ ($A$ is any of the hydrodynamic parameters) and neglecting terms of order higher than the first in $\epsilon^1$, $\rho_2^1$, $v_1^1$ and $v_2^1$ in [11]–[14], we obtain the linearized shock conditions as follows:

$$D\epsilon^1 + \epsilon_1^0 v_1^1 = 0, \qquad [18]$$

$$\rho_2^0 D\epsilon^1 + \epsilon_2^0 D\rho_2^1 - \rho_2^0 \epsilon_2^0 v_2^1 = 0, \qquad [19]$$

$$(\rho_1 - \rho_2^0)D^2\epsilon^1 - (c^2 + \epsilon_2^0 D^2)\rho_2^1 + 2\rho_1\epsilon_1^0 Dv_1^1 + 2\rho_2^0\epsilon_2^0 Dv_2^1 = 0 \qquad [20]$$

and

$$c_0^2\rho_2^1 - \rho_1(E(\epsilon_2^0) + 1)Dv_1^1 + \rho_1 E^0 Dv_2^1 = 0, \qquad [21]$$

where $c_0$ is the sound speed in the pure phase "2" in front of the shock, so that

$$c_0^2 = \left(\frac{\mathrm{d}p_2}{\mathrm{d}\rho_2}\right)_{\rho_2 = \rho_2^0}. \qquad [22]$$

The condition of solvability of the linear system [18]–[21] is

$$\begin{vmatrix} D & 0 & \epsilon_1^0 & 0 \\ \rho_2^0 D & \epsilon_2^0 D & 0 & -\rho_2^0\epsilon_2^0 \\ (\rho_1 - \rho_2^0)D^2 & -(c_0^2 + \epsilon_2^0 D^2) & 2\rho_1\epsilon_1^0 D & 2\rho_2\epsilon_2^0 D \\ 0 & c_0^2 & -\rho_1(E^0 + 1)D & \rho_1 E^0 D \end{vmatrix} = 0. \qquad [23]$$

Apart from the trivial double solution, $D^2 = 0$, corresponding to convection of concentration disturbances by the dispersed phase, we immediately obtain the speed of the weak shock ("speed of sound" in the considered two-phase media) in the form

$$D^2 = c_0^2 \frac{\rho_2^0}{\rho_1\epsilon_2^0} \frac{\rho_1((\epsilon_2^0)^2 + E^0) + \rho_2^0\epsilon_1^0\epsilon_2^0}{\rho_1\epsilon_1^0 E^0 + \rho_2^0\epsilon_2^0(E^0 + 1)}. \qquad [24]$$

Obviously [24] gives the velocities for two waves propagating to the "left" and to the "right" in the chosen coordinate system. It is identical with the wave speed predicted by Geurst (1985a,b) in the limit where the relative velocity ahead of the wave is small.

It is useful to obtain from [24] the speed of propagation of a small-amplitude wave for some important extremes.

At $\epsilon_1 \to 0 (\epsilon_2 \to 1)$ the second phase occupies the whole space and from [24] we obtain (as should be expected)

$$D^2 = c_0^2.$$

In the case when the density of the dispersed phase is much lower than that of the continuous phase (this case corresponds to the most straightforward application of the considered model, i.e. to a bubble–liquid mixture), the weak shock propagation speed is as follows:

$$D^2 = c_0^2 \frac{\rho_2^0}{\rho_1} \frac{(\epsilon_2^0)^2 + E^0}{\epsilon_1^0 \epsilon_2^0 E^0}. \qquad [25]$$

This wave speed is real because $E$ is proportional to the difference between the mean square of the fluid velocity and the square of the mean fluid velocity in a coordinate system relative to the dispersal phase and must always be positive (Wallis 1989b); put another way, the kinetic energy due to relative motion must be positive.

When Maxwell's approximate expression for the exertia (Maxwell 1881; Wallis 1989c, 1991) is used,

$$E = \frac{\epsilon_2}{2}, \qquad [26]$$

the expression for $D$ reduces in the general case to

$$D^2 = c_0^2 \frac{\rho_2^0}{\rho_1 \epsilon_2^0} \frac{\rho_1 (2\epsilon_2^0 + 1) + 2\rho_2^0 \epsilon_1^0}{\rho^1 \epsilon_1^0 + \rho_2^0 (\epsilon_2^0 + 2)} \qquad [27]$$

and in the case of bubble–liquid flow to

$$D^2 = c_0^2 \frac{\rho_2^0}{\rho_1} \frac{2\epsilon_2^0 + 1}{\epsilon_1^0 \epsilon_2^0}. \qquad [28]$$

We note that in the last case, the dependence of the wave speed on the concentration has a minimum at

$$\epsilon_2^* = \frac{\sqrt{3} - 1}{2} \simeq 0.37. \qquad [29]$$

The corresponding minimum value of the wave speed is

$$\frac{D_{\min}^2}{c_0^2} \simeq 7.46 \frac{\rho_2^0}{\rho_1}. \qquad [30]$$

These results show a considerable reduction in the speed of the weak shock (or the "speed of sound") in the two-phase mixture due to inertial coupling of the phases. An estimation of the influence of the inertial interphase interaction on the wave speed can be made with help of the next simple example. We consider the system of air bubbles in water at standard conditions, so that $\rho_2 \ll \rho_1$. Assuming that $\epsilon_1^0 = \epsilon_2^0 = 0.5$, for the Maxwell exertia, we immediately find from [28] that $D \simeq 29.5$ m/s. Without an interphase interaction $E = 0$ [i.e. in the case of a *stratified compressibility wave* in accordance with Wallis (1969)], we find from [24]

$$D^2 = c_0^2 \frac{\rho_1 \epsilon_2^0 + \rho_2^0 \epsilon_1^0}{\rho_1 \epsilon_2^0}. \qquad [31]$$

When $\rho_2 \ll \rho_1$ this gives $D^2 = c_0^2$, indicating that inertial interphase interaction leads to a considerable decrease in the wave speed.

An opposite extreme is the "homogeneous" wave speed, i.e. the velocity of wave propagation in the suspension with extremely strong interphase interaction when there is no relative motion of the phases. To obtain the wave speed in such a case we assume $E \to \infty$ in [24], so that

$$D^2 = c_0^2 \frac{\rho_2^0}{\epsilon_2^0} \frac{1}{\rho_1 \epsilon_1^0 + \rho_2^0 \epsilon_2^0}. \qquad [32]$$

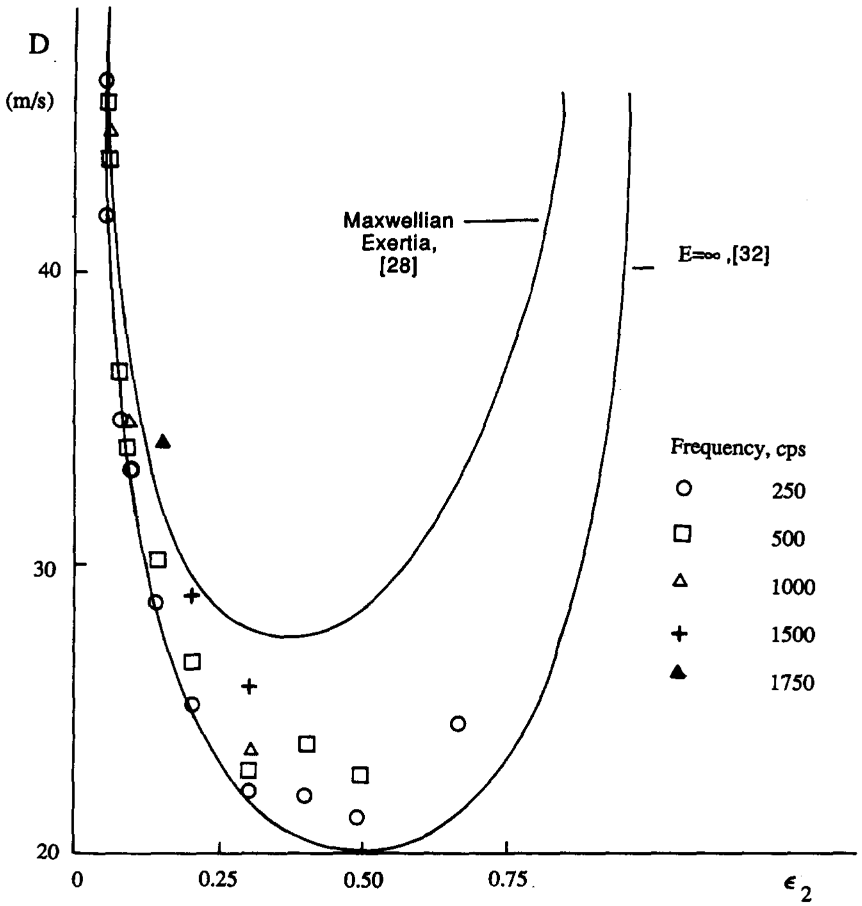


Figure 1. Wave speed in an air–water mixture vs void fraction at isothermal conditions: $p = 1$ atm, $T = 20°C$; comparison with Karplus (1958) data.

To assess the effect of inertial interphase interaction on wave propagation we compare, at $\rho_2^0 \ll \rho_1$, the result [28] both with the wave speed for a homogeneous suspension and the experimental data by Karplus (1958). The result of such a comparison is represented in figure 1 for an air–water mixture at normal conditions in the case of an isothermal compression of the bubbles.

It is clearly seen from the curves that an interphase interaction limited to inertial coupling leads to an increase in the wave speed compared with the case of the "extremely strong" interphase interaction corresponding to an absence of relative motion of the phases (i.e. $E \to \infty$ in the model under consideration). While the lower curve representing the wave speed in a homogeneous suspension is symmetrical with the minimum at $\epsilon_2^0 = \epsilon_1^0 = 0.5$, the curve for the wave speed in an inertially coupled mixture is clearly shifted to the left with the minimum at $\epsilon_2^0 \simeq 0.37$, as was pointed out in [29]. While at a low concentration of the dispersed phase both wave speeds practically coincide up to $\epsilon_1^0 \simeq 0.1$, at high concentration the results show a pronounced difference in $D$ for both cases; e.g. at $\epsilon_2^0 = 0.8$ for a homogeneous suspension we find $D \simeq 62.8$ m/s, while for inertially coupled phases $D \simeq 163$ m/s. Within the range of "intermediate" concentrations ($\epsilon_2^0 = 0.25$ to 0.6) the relative difference of the wave speeds for both cases is from 20 to 50%.

As may be seen in figure 1, the experimental data by Karplus (1958) actually lie between the two curves. While the experimental data obtained for low frequencies of the included disturbances are somewhat closer to the lower curve corresponding to the homogeneous mixture (especially in the range of low concentrations of the dispersed phase), experimental data for higher frequencies are clearly shifted to the direction of the curve obtained for inertially coupled phases. This is to be expected because inertia effects become dominant at high frequencies, while drag forces ensure homogeneous motion at low frequencies. If drag forces are added to [3] and [4], the propagation of disturbances becomes dispersive with the wave speed at low frequency, given by [32], and at high frequency, given by [27]. There is evidence for this trend in figure 1. For a more complete description, thermal relaxation would also have to be considered.

We emphasize that the shock waves considered above are actually due to the combination of two phenomena, i.e. the jump of the concentration of the dispersed phase across the shock *and* the jump of the density of the dispersed compressible phase across the shock. The relationship between the jumps of gas density and dispersed phase concentration across the weak shock can be obtained from [23] as follows:

$$\rho_2^1 = -\frac{\rho_2^0(\epsilon_2^0 + E^0)}{\epsilon_1^0\epsilon_2^0}\frac{\rho_1((\epsilon_2^0)^2 + E^0) + \rho_2^0\epsilon_1^0\epsilon_2^0}{(\rho_1 + \rho_2^0)\epsilon_1^0\epsilon_2^0E^0 + (\rho_1 + \rho_2^0)(\epsilon_2^0)^2E^0 + \rho_1(E^0)^2 + \rho_2^0(\epsilon_2^0)^2}\epsilon^1. \qquad [33]$$

In the case of bubble–liquid flow [33] reduces to

$$\rho_2^1 = -\rho_2^0\frac{(\epsilon_2^0)^2 + E^0}{\epsilon_1^0\epsilon_2^0E^0}\epsilon^1. \qquad [34]$$

As can be seen from [33] and [34], at least for the small-amplitude shock, the compression jump of the mass density of the dispersed phase is necessarily associated with the "rarefaction" jump of the concentration $\epsilon_2$ of the dispersed phase (i.e. with the voidage jump corresponding to the decrease of $\epsilon_2$ across the shock), and vice versa—the rarefaction jump of the density $\rho_2$ is associated with the "compression" jump of the concentration $\epsilon_2$.

For a gas–liquid mixture the relationship [34] between the small-amplitude jumps of the density and the concentration of the dispersed phase, with the help of [25], can now be rewritten in the simple form

$$\rho_2^1 = -\rho_1\frac{D^2}{c_0^2}\epsilon^1. \qquad [35]$$

In the general case the relationship is as follows:

$$\rho_2^1 = -\frac{\rho_1\rho_2^0(\epsilon_2^0 + E^0)D^2}{\epsilon_1^0\epsilon_2^0(\rho_2^0c_0^2 + \rho_1E^0D^2)}\epsilon^1. \qquad [36]$$

Equations [35] and [36] give the relationships between the small-amplitude jumps of the voidage, density and the propagation speed of the wave.

Now we analyze the velocities of the fluid and the dispersed phase behind the small-amplitude shock. From [17] and [18], for the fluid velocity it immediately follows that

$$v_1 = -D\left(1 + \frac{\epsilon^1}{\epsilon_1^0}\right). \qquad [37]$$

To analyze the jump of the velocity of the dispersed phase we start from the case of bubble–liquid flow. From [17], [19] and the relationship between the jumps of the gas density and the voidage, [34], we obtain

$$v_2 = -D\left\{1 + \epsilon^1\left(\frac{(\epsilon_2^0)^2 + E^0}{\epsilon_1^0\epsilon_2^0E^0} - 1\right)\right\}. \qquad [38]$$

It can easily be shown that for any positive $E^0$

$$\frac{(\epsilon_2^0)^2 + E^0}{\epsilon_1^0\epsilon_2^0E^0} - 1 > 0. \qquad [39]$$

The results in [37]–[39] imply that

$$|v_1| > D, \quad |v_2| > D \quad \text{at} \quad \epsilon^1 > 0(\rho_2^1 < 0) \qquad [40a]$$

and

$$|v_1| < D, \quad |v_2| < D \quad \text{at} \quad \epsilon^1 < 0(\rho_2^1 > 0) \qquad [40b]$$

behind the front of the wave, so that both the velocity of the fluid and the velocity of the dispersed phase exceed the "sound" speed of the two-phase dispersion $D$ in the case of the "compression" shock of the concentration associated with the rarefaction shock of the gas density. Naturally, the flow of both phases is "subsonic" (relative to $D$) behind the shock in the case of the "rarefaction" shock of the concentration associated with the compression shock of the gas density.

In a pure gas the flow approaching a compression shock is supersonic, but this result cannot be derived from [33]–[36] and [38] because these relationships fail at $\epsilon_1^0 = 0$ ($\epsilon_2^0 = 1$) due to the neglect of $\rho_2$ compared with $\rho_1$ in using [25] instead of [24].

The result [40a, b] is still valid in the general case when $\rho_2$ is not necessarily negligible compared to $\rho_1$. In the general case, the formula [37] for the fluid velocity behind the shock is valid as well. The jump of the velocity of the dispersed phase $v_2^1$ can be obtained from [19] as

$$v_2^1 = \frac{D}{\rho_2^0 \epsilon_2^0}(\rho_2^0 \epsilon^1 + \epsilon_2^0 \rho_2^1). \qquad [41]$$

Substituting the relationship [33] between the jumps of the density $\rho_2^1$ and the voidage $\epsilon^1$ into [41], after rather lengthy but straightforward algebra it can be proved that $v_2^1 = -C(\epsilon_2^0)\epsilon^1$, where $C(\epsilon_2^0) > 0$ within the interval $0 \leqslant \epsilon_2^0 \leqslant 1$, so that the result [40a, b] on the character of the velocity of the dispersed phase obtained for the bubble–liquid dispersion is valid in the general case.

If the compressibility of at least one of the phases is not taken into account, the governing equations [1]–[5] do not allow a solution in the form of the concentration shock propagating relatively to the dispersed phase (the details of the corresponding derivation are omitted here). The last circumstance is connected with the neglect of interparticle interaction. An analysis of this situation for gas–solid mixtures has been given by Sergeev (1988/89).

The results obtained show that for physically realistic situations the relative velocity between the phases ahead of the wave can usually be neglected. Indeed, the wave velocity at usual conditions is of the order of tens of m/s. Bubble–liquid flows with relative velocities of the phases with this order of magnitude can hardly be imagined.

## REFERENCES

GEURST, J. A. 1985a Two-fluid hydrodynamics of bubbly liquid/vapor mixture including phase change. *Philips J. Res.* **40,** 352–374.

GEURST, J. A. 1985b Virtual mass in two-phase bubbly flow. *Physica* **129A,** 233–261.

GEURST, J. A. 1986 Variational principles and two-fluid hydrodynamics of bubbly liquid/gas mixtures. *Physica* **135A,** 455–486.

KARPLUS, H. B. 1958 The velocity of sound in a liquid containing gas bubbles. USAEC Report COO-248.

MAXWELL, J. C. 1881 In *A Treatise on Electricity and Magnetism*, 2nd edn, p. 388. Clarendon Press, Oxford.

NIGMATULIN, R. I. 1978 *Fundamentals of the Mechanics of Heterogenous Media.* Nauka, Moscow (in Russian).

SERGEEV, Y. A. 1988/89 Propagation of waves and dispersed-phase concentration discontinuities in a gas-fluidized bed of weakly interacting particles. *Fluid Dynam.* **23,** 706–713.

WALLIS, G. B. 1969 *One-dimensional Two-phase Flow.* McGraw-Hill, New York.

WALLIS, G. B. 1989a On Geurst's equations for inertial coupling in two-phase flow. Presented at a *Wkshp*, Inst. for Mathematics and its Applications, Univ. of Minnesota, 1989. In *Two Phase Flow and Waves* (Edited by JOSEPH, D. D. & SCHAEFFER, D. G.). Springer, New York.

WALLIS, G. B. 1989b Two-phase potential flow. Workshop: constitutive relationships and models in continuum theories of multiphase flows. *NASA Conf. Publ.* **3047,** 19–34.

WALLIS, G. B. 1989c Inertial coupling in two-phase flow: macroscopic properties of suspensions in an inviscid fluid. *Multiphase Sci. Technol.* **5,** 239–361.

WALLIS, G. B. 1991 The averaged Bernoulli equation and macroscopic equations of motion for the potential flow of a two-phase dispersion. *Int. J. Multiphase Flow* **17,** 683–695.